数理解析研究所講究録
983 巻 1997 年 70-75

THE EXISTENCE AND THE CONTINUATION OF HOLOMORPHIC SOLUTIONS FOR CONVOLUTION EQUATIONS IN A HALF-SPACE IN $\mathbf{C}^n$

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ABSTRACT. — We study holomorphic solutions for convolution equations (E) $T*f=g$ in a half-space in $\mathbf{C}^n$. Under a natural condition (the condition (S)), we will prove the existence of solutions of (E) and the analytic continuation of homogeneous equation (E$'$) $T*f=0$.

1. INTRODUCTION

Let Ω be a convex domain and let K be a compact convex set in $\mathbf{C}^n$. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω provided with the topology of uniform convergence on compact subsets of Ω, and by $\mathcal{O}(K)$ the space of germs of functions holomorphic on

K, endowed with the usual topology of the inductive limit. Then each nonzero analytic functional $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K (or equivalently, $T \in \mathcal{O}'(K)$) defines a continuous linear convolution operator

$$T* \ : \ \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

which is given by

$$(T * f)(z) = T_\zeta(f(z+\zeta)), \ z \in \Omega.$$

If $K = \{0\}$, the convolution operator $T*$ is a linear partial differential operator of infinite order with constant coefficients on $\mathcal{O}(\Omega)$. The convolution equation has been historically studied by many authors, especially the equation in the category of holomorphic functions defined on a complex domain. For example, using the notion of an entire function of completely regular growth on a fixed ray, Morzhakov [1] established sufficient condition for $T*$ to be surjective in the general case, and gave a criterion for the solvability for three classes of domains: smooth domains, products of smooth planar domains, and domains whose boundaries consist of smooth points and vertices. On the other hand, under the condition (S) due to Kawai [2], Ishimura - Y. Okada [3] studied the existence and the continuation problem of holomorphic solutions for convolution equations of hyperfunction kernels in the tube domain. In [4], Ishimura and the author proved that the property of completely regular growth is equivalent to the condition (S) for entire functions, in more general case, *i.e.* for sub-harmonic functions.

In this paper, we consider the convolution equation in the case where Ω is a half-space, and under the condition $(\mathrm{S})_{\zeta_0}$, we will prove the existence of solutions of (E) and the analytic continuation of homogeneous equation (E$'$).

Most of results is based on the paper [3], and refer to it for the details of proofs.

2. PRELIMINARIES

Let

$$|z|^2 = z_1\overline{z_1} + \cdots + z_n\overline{z_n}, \qquad \langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

for

$$z = (z_1, \cdots, z_n), \ w = (w_1, \cdots, w_n) \in \mathbf{C}^n.$$

For $\zeta_0 \in \mathbf{C}^n$ and $|\zeta_0| = 1$, we put

$$\Omega = \{ \ \zeta \in \mathbf{C}^n \mid \mathrm{Re}\langle \zeta_0, \zeta\rangle < 0 \ \}$$

and take a compact convex set K as $K \subset \Omega$, *i.e.* $\Omega + K = \Omega$. As it is well-known, the properties of convolution operator are reflected in the properties of the Fourier-Borel transform of T, namely

$$\widehat{T}(\zeta) = T_z(\exp\langle z, \zeta\rangle),$$

which is an entire function of exponential type satisfying the following estimate:

Theorem 2.1.(Polyà-Ehrenpreis-Martineau) If $T \in \mathcal{O}'(\mathbf{C}^n)$ and T is carried by a compact set $K \subset \mathbf{C}^n$, then $\widehat{T}(\zeta)$ is an entire function and for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$|\widehat{T}(\zeta)| \leq C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|), \quad \zeta \in \mathbf{C}^n \tag{2.1}$$

where $H_K(\zeta) = \sup\limits_{z \in K} \mathrm{Re}\langle z, \zeta\rangle$ is the supporting function of K.
Conversely, if K is a compact convex set and $f(\zeta)$ an entire function satisfying (2.1) for every $\varepsilon > 0$, there exists an analytic functional $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K such that $\widehat{T}(\zeta) = f(\zeta)$.

In this paper, we suppose the following condition for the entire function $\widehat{T}(\zeta)$.

Definition 2.2. We say that $\widehat{T}(\zeta)$ satisfies the condition (S) to the direction ζ_0 or simply it satisfies the condition $(\mathrm{S})_{\zeta_0}$ if we have

$$\begin{cases} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } r \in \mathbf{R} \text{ with } r > N, \\ \text{we can find } \zeta \in \mathbf{C}^n, \text{ which satisfies} \\ |\zeta - \zeta_0| < \varepsilon r, \\ |\widehat{T}(\zeta)| \geq \exp(-\varepsilon r). \end{cases}$$

3. THE EXISTENCE OF HOLOMORPHIC SOLUTIONS

We will prove the surjectivity theorem under the condition $(\mathrm{S})_{\zeta_0}$.

Theorem 3.1. Let $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K. Assume that $\widehat{T}(\zeta)$ satisfies the condition $(\mathrm{S})_{\zeta_0}$. Then the convolution operator

$$T* \;:\; \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

is surjective.

Proof. The transpose of

$$P = T* \ : \ \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

is

$${}^tP = \check{T}* \ : \ \mathcal{O}'(\Omega) \longrightarrow \mathcal{O}'(\Omega + K)$$

with $\widehat{\check{T}}(\zeta) = \widehat{T}(-\zeta)$. By the standard argument, it is enough to prove that ${}^tP$ is injective and that the image of ${}^tP$ is weakly closed. In fact, the injectivity of ${}^tP$ shows that the image of P is dense, and the closedness of the image of ${}^tP$ shows the closedness of the image of P. Because T is not 0, the injectivity of ${}^tP$ is clear. Then we will show that the image of ${}^tP$ is weakly closed. To do this, we use the following division lemma, which we can prove in an analoguous way to the proof of Lemma 2.1 in [3].

Lemma 3.2. Let f, g and h be entire functions satisfying $fg = h$, and K and L be two compact convex sets in $\mathbf{C}^n$ with $K,\ L \subset \Omega$. We suppose that for every $\varepsilon > 0$, f and h satisfy the following estimates with constants $A_\varepsilon > 0$ and $B_\varepsilon > 0$,

$$\begin{cases} \log|f(\zeta)| \leq A_\varepsilon + H_K(\zeta) + \varepsilon|\zeta|, \\ \\ \log|h(\zeta)| \leq B_\varepsilon + H_L(\zeta) + \varepsilon|\zeta|, \end{cases}$$

for any $\zeta \in \mathbf{C}^n$. We also assume that f satisfies the condition $(\mathrm{S})_{\zeta_0}$. Then for any $\varepsilon > 0$, there exists a compact convex set $M = M_\varepsilon \subset \mathbf{C}^n$ and $C_\varepsilon > 0$ such that

$$\begin{cases} M \subset \Omega \\ \\ \log|g(\zeta)| \leq C_\varepsilon + H_M(\zeta). \end{cases}$$

End of the proof of the theorem. — Let $\{T_\nu\}$ be a sequence in $\mathcal{O}'(\Omega)$ and assume that $\{{}^tPT_\nu\}$ converges to $S \in \mathcal{O}'(\Omega + K)$ in $\mathcal{O}'(\Omega + K)$. By taking the Fourier-Borel transform, $\widehat{T}(-\zeta)\widehat{T}_\nu(\zeta)$ converges to $\widehat{S}(\zeta)$. Then it is well-known that $G(\zeta) = \dfrac{\widehat{S}(\zeta)}{\widehat{T}(-\zeta)}$ becomes an entire function. By Lemma 3.2 and Theorem 2.1, there exists a compact convex set M and $\mu \in \mathcal{O}'(\mathbf{C}^n)$ carried by M such that $\widehat{\mu}(\zeta) = G(\zeta)$ and ${}^tP\mu = \check{T} * \mu = S$, *i.e.* $S \in \operatorname{Im}{}^tP$. □

4. THE CHARACTERISTIC SET AND THE CONTINUATION OF HOMOGENEOUS SOLUTIONS

Under the condition $(\mathrm{S})_{\zeta_0}$, we shall now solve the problem of continuation for the solutions of the homogeneous equation (E$'$). For any open set $U \subset \mathbf{C}^n$, we set:

$$\mathcal{N}(U) = \{ f \in \mathcal{O}(U) \mid T * f = 0 \}.$$

For an open set $V \subset \mathbf{C}^n$ with $U \subset V$, the problem is formulated as to get the condition so that the restriction map

$$r : \mathcal{N}(V) \longrightarrow \mathcal{N}(U)$$

is surjective.

In order to describe the theorem of continuation, we will prepare the notion of characteristics which is a natural generalization of the case of usual differential operators of finite order with constant coefficients. We define the sphere at infinity S^{2n-1}_∞ by $(\mathbf{C}^n \setminus \{0\})/\mathbf{R}_+$ and consider the compactification with directions $\mathbf{D}^{2n} = \mathbf{C}^n \sqcup S^{2n-1}_\infty$ of $\mathbf{C}^n$. For $\zeta \in \mathbf{C}^n \setminus \{0\}$, we denote by $\zeta\infty \in S^{2n-1}_\infty$ the equivalence class of ζ, *i.e.*

$$\{\zeta\infty\} = (\text{the closure of } \{ t\zeta \mid t > 0 \} \text{ in } \mathbf{D}^{2n}) \cap S^{2n-1}_\infty.$$

For $\varepsilon > 0$, we put:

$$\begin{cases} V_{\widehat{T}}(\varepsilon) = \{ \zeta \in \mathbf{C}^n \mid \exp(\varepsilon|\zeta|)|\widehat{T}(\zeta)| < 1 \}, \\ W_{\widehat{T}}(\varepsilon) = (\text{the closure of } V_{\widehat{T}}(\varepsilon) \text{ in } \mathbf{D}^{2n}) \cap S^{2n-1}_\infty. \end{cases}$$

Now we define the characteristic set of $T*$.

Definition 4.1. With the above notation, we define the characteristics of $T*$ (at infinity)

$$\mathrm{Char}_\infty(T*) = \text{the closure of } \bigcup_{\varepsilon>0} W_{\widehat{T}}(\varepsilon).$$

Under the above situation, we can state the theorem of the continuation without proof.

Theorem 4.2. Let $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K and $f \in \mathcal{O}(\Omega + K)$ be a solution of $T * f = 0$. Assume that $\widehat{T}(\zeta)$ satisfies the condition $(\mathrm{S})_{\zeta_0}$. If $\zeta_0 \notin \mathrm{Char}_\infty(T*)$, then the restriction map

$$r : \mathcal{N}(\mathbf{C}^n) \longrightarrow \mathcal{N}(\Omega + K)$$

is surjective, that is, f can be analytically continued to the whole $\mathbf{C}^n$.

References

[1] V. V. Morzhakov : *Convolution equations in convex domains of* $\mathbf{C}^n$, Compl. Anal. and Appl. '87, Sofia, pp.360–364 (1989).
[2] T. Kawai : *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **17**, pp.467–517 (1970).
[3] R. Ishimura and Y. Okada : *The existence and the continuation of holomorphic solutions for convolution equations in tube domains*, Bull. Soc. Math. France, **122**, pp.413–433 (1994).
[4] R. Ishimura et J. Okada : *Sur la condition (S) de kawai et la propriété de croissance régulière d'une fonction sous-harmonique et d'une fonction entière*, Kyushu J. Math. **48**, pp.257–263 (1994).